Sets, Set Operations, and Cardinality

What is a Set?

A set is a collection of objects, called elements, without repetition. We use curley braces, $\{\}$, to denote them.

ex $A = \{1, 2, 3\}$ A is the set containing 1, 2, and 3

$B = \{\text{fish, cats, dogs}\}$ B is the set of the top 3 most popular pets in the US

$\text{fish} \in B$ fish is an element of B

One special set we deal with is the set that contains no elements. We call this the empty set and we denote it with $\emptyset$.

$$\text{empty set} = \{ \; \} = \emptyset$$

If all the elements are contained in another set we say the first is a subset of the larger second.

$\{1,2\} \subseteq \{1,2,3\}$ The set containing 1 and 2 is a subset of the set containing 1, 2 and 3

Set Builder Notation

Rather than listing all elements of a set we can describe them or give conditions that they satisfy. Set Builder notation is the standard way that we do this.

$$\left\{ \text{named element} : \text{conditions that any element in the set will satisfy} \right\}$$

ex $A = \{ x : x \text{ is an integer and } 3 \leq x < 10 \}$

The only integers greater than 3 and less than 10 are 3, 4, 5, 6, 7, 8, 9 so

$$A = \{3, 4, 5, 6, 7, 8, 9\}$$

$$B = \{ (\text{roll } 1, \text{roll } 2) : \text{rolls of 2 6-sixed die add to 5} \}$$

$$B = \{ (1,4), (2,3), (3,2), (4,1) \}$$

Given a long list of elements in a set

$\{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97\}$

we can use set builder notation to more concisely express it.

$$\{x : x \text{ is prime and less than } 100\}$$

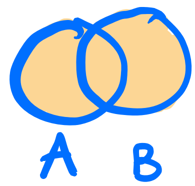
Set Operations

Set operations allow us to make new sets from other sets. We'll cover

- union $\cup$
- intersection $\cap$
- complement A', A^c
- cross product $\times$

Union

The union of two sets A and B, $A \cup B$, is a larger set that contains everything in A and everything in B.

- $A \cup B = \{x : x \text{ is in } A \text{ or } x \text{ is in } B\}$
- Visually $A \cup B$ 

- A bakery has 3 kinds of cookies: sugar, chocalate chip, peanut butter. And 2 kinds of cakes: lemon and carrot.

  $O = \{\text{sugar, chocalate chip, peanut butter}\}$

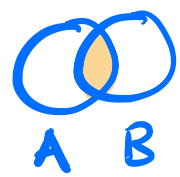
  $A = \{\text{lemon, carrot}\}$

$O \cup A$ is the different kinds of baked goods available

Intersection

The intersection of sets A and B is their overlap. It contains the elements that are in both and and B.

- $A \cap B = \{x : x \text{ is in } A \text{ and is in } B\}$
- Visually $A \cap B$



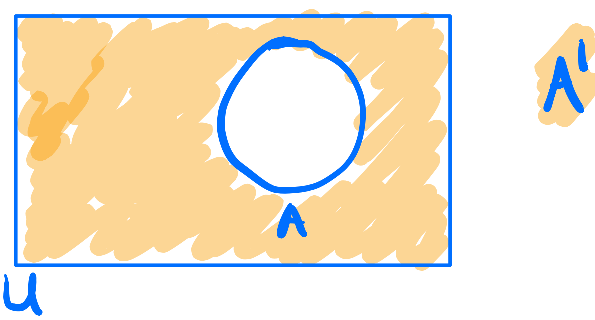
- If A was set of all classes your major requires and I was set of all classes your minor requires,

Then $A \cap I$ is the set of classes that satisfy both your major and minor

Complement

The complement of a set is everything not in that set. We often restrict this to everything not in that set <u>but</u> still in some given universal set.

- Given U. Then
  $$A' = \{x : x \text{ is not in } A \text{ and } x \text{ is in } U\}$$
- Visually

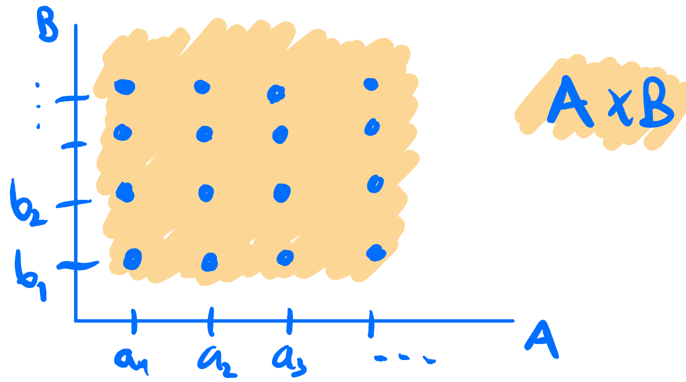


- If U was all meals at a restaurant and M was all meals containing meat, Then M' are all vegetarian meals at restaurant

Cartesian Product

The cartesian product of A and B ($A \times B$) is the set of ordered pairs where the first element is in A and the second is in B.

- $A \times B = \{(x,y) : x \text{ is in } A \text{ and } y \text{ is in } B\}$
- Visually



- If at an ice cream shop S corresponded to sizes that can be ordered and F corresponded to flavors that could be ordered, Then $S \times F$ corresponds to valid ice cream orders.

Cardinality

The number of elements in a set is called its cardinality, symbolized by $n(A)$ or $|A|$

ex $n(\{1,5,7\})=3$

Consider $n(A \cup B)$ if you know $A=\{1,2,3,4\}$ and $B=\{1,3,5\}$

then you could find $A \cup B=\{1,2,3,4,5\}$

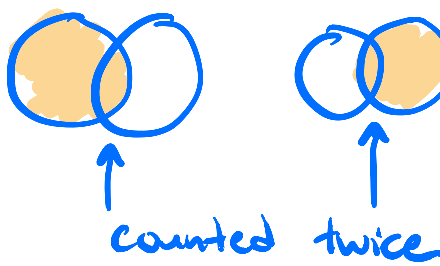
$$n(A \cup B)=5$$

But without being told the contents of A and B you could answer it by using various cardinality relations:

- $n(A \cup B)=n(A)+n(B)-n(A \cap B)$
- $n(A')=n(U)-n(A)$
- $n(A \times B)=n(A) * n(B)$

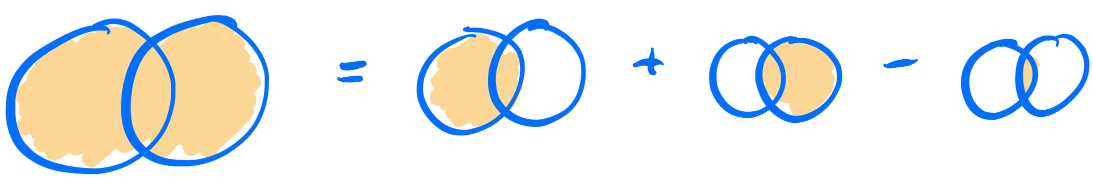
Why is $n(A \cup B)$ not $n(A) + n(B)$?

Because A and B might share, or have overlapping, elements. If they did then adding $n(A)$ and $n(B)$ double counts these elements.



To get rid of this over count, we subtract the size of its overlapping region, $n(A \cap B)$

So $$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$



Rearranging gives another 3 relations

$$n(A \cap B) = n(A) + n(B) - n(A \cup B)$$
$$n(A) = n(A \cup B) + n(A \cap B) - n(B)$$
$$n(B) = n(A \cup B) + n(A \cap B) - n(A)$$