$$A = \begin{bmatrix} 5 & 3 & 2 \\ 1 & 0 & 5 \\ -1 & 2 & 3 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B_1 = \begin{bmatrix} 5 \\ 3 \\ 2 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$$

Solve for X_i in the following:

$$AX_1 = B_1, \quad AX_2 = B_2, \quad AX_3 = B_3$$

$$\left[\begin{array}{ccc|c} 5 & 3 & 2 & 5 \\ 1 & 0 & 5 & 3 \\ -1 & 2 & 3 & 2 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 5 & 3 & 2 & -1 \\ 1 & 0 & 5 & 0 \\ -1 & 2 & 3 & -1 \end{array}\right] \qquad \vdots$$

$\hookrightarrow \dots \qquad \hookrightarrow \dots$

$$5x = 15 \xrightarrow{1/5} x = 5$$

$$AX = B \xrightarrow[?]{?} X = ?$$

Multiplicative inverses:

- for numbers $\quad n \to \frac{1}{n} \qquad n \cdot \frac{1}{n} = 1$
- for matrices $\quad M \to M^{-1} \qquad M \cdot M^{-1} = I$

Identity Matrix

The Identity matrix is a square matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}$$

$$2\times 2 \qquad 3\times 3 \qquad n\times n$$

$$A=\begin{bmatrix} 5 & 3 & 2 \\ 1 & 0 & 5 \\ -1 & 2 & 3 \end{bmatrix}, \quad X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B_1=\begin{bmatrix} 5 \\ 3 \\ 2 \end{bmatrix}, \quad B_2=\begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}, \quad B_3=\begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$$

Solve for X_i in the following:

$$AX_1=B_1, \quad AX_2=B_2, \quad AX_3=B_3$$

If we can find A^{-1} we can easily solve the rest

$$X_1=A^{-1}\cdot B_1 \qquad X_2=A^{-1}B_2 \qquad X_3=A^{-1}B_3$$

Do All Matrices Have Inverses? No

- If M is not square it's not invertible
- If the <u>determinant</u> of M is 0 then it has no inverse
- We'll define a square matrix to be <u>singular</u> if it has no inverse

<u>Determinant of 2x2</u>

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\det(M) = ad - bc$$

*If the determinant of a matrix is <u>not</u> 0 it <u>does</u> have an inverse

<u>ex</u> $\det\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = 4 - 6 = -2$

$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ does have an inverse

Inverse of a 2x2 Matrix

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$M^{-1} = \frac{1}{\underbrace{ad - bc}_{\text{determinant of } M}} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

- swap the main diagonal
- negate the weak diagonal

ex

$$A = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} \quad A^{-1} = \frac{1}{2 \cdot 3 - 5 \cdot 1} \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$$

$$= 1 \cdot \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$$

$$AA^{-1} = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 6-5 & -10+10 \\ 3-3 & -5+6 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} \qquad A^{-1} = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$$

$$\begin{matrix} 2x + 5y = 3 \\ x + 3y = 1 \end{matrix} \longrightarrow \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}}_{I}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 9-5 \\ -3+2 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$$

$$\begin{matrix} 2x + 5y = 3 \\ x + 3y = 1 \end{matrix} \longrightarrow \begin{matrix} x = 4 \\ y = -1 \end{matrix}$$

Inverses of Matrices with dimensions $>2\times 2$

Need to know

- how to check if a given matrix B is the inverse of A
- how to use the inverse to solve a system

The inverse of

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ 2 & 1 & 2 \end{pmatrix}$$

is either

$$\begin{pmatrix} 1 & 1 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 1 & 0 \\ -2 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Find it and use it to solve the following system:

$$\begin{aligned} x + \quad z &= 2 \\ -z &= 4 \\ 2x + y + 2z &= 3 \end{aligned}$$

The inverse of

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ 2 & 1 & 2 \end{pmatrix} = A$$

is either

$$\underbrace{\begin{pmatrix} 1 & 1 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}}_{=B} \quad \text{or} \quad \underbrace{\begin{pmatrix} 1 & 1 & 0 \\ -2 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}}_{=C}.$$

- $A \cdot A^{-1} = I$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ 2 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1+0-1 & & \\ & & \\ & & \end{pmatrix} = \begin{pmatrix} 0 & & \\ & & \\ & & \end{pmatrix}$$

$$B \neq A^{-1}$$

- $AC = I$? $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ 2 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ -2 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

So if $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ 2 & 1 & 2 \end{pmatrix}$ then $A^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ -2 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$.

Find it and use it to solve the following system:

$$\begin{aligned} x + \quad z &= 2 \\ -z &= 4 \\ 2x + y + 2z &= 3 \end{aligned}$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ 2 & 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1}\begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ -2 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 2+4 \\ -4+3 \\ -4 \end{pmatrix} = \begin{pmatrix} 6 \\ -1 \\ -4 \end{pmatrix}$$

$$x = 6$$
$$y = -1$$
$$z = -4$$