4.1 and 4.2 Matrix Algebra

A matrix is an array or table of numbers or variables.

An $m \times n$ matrix has m rows and n columns

$$C = \begin{bmatrix} 5 & 7 & 11 \\ 13 & 12 & 9 \end{bmatrix}$$ has 2 rows and 3 colums, it is 2×3 matrix

We can specify a particular entry in a matrix by giving its row and column.

$C_{1,2}$ is 7 $$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \end{bmatrix}$$

In general M_{ij} refers to the entry in the i-th row and j-th column of M.

Matrices with certain dimensions might be given a special name.

- A matrix having only 1 row is a row matrix

$$A = \begin{bmatrix} 1 & 3 & 5 & 1 \end{bmatrix}$$ is a 1×4 row matrix

- A matrix having only 1 col is a column matrix

$$B = \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}$$ is a 3×1 col matrix

- A matrix with equal numbers of rows and cols is a square matrix

$$C = \begin{bmatrix} 1 & 2 & 4 \\ 0 & -1 & 0 \\ 5 & 7 & 1.1 \end{bmatrix}$$ is a 3×3 square matrix

Two matrices are equal if they

- have the same dimensions
- corresponding entries are equal

If $A = B$ $a_{ij} = b_{ij}$ for all rows i and columns j

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \neq \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$$

<u>ex</u>

$$A = \begin{bmatrix} 5x & -z \\ 3x+y & y \end{bmatrix} \qquad B = \begin{bmatrix} 5 & 7 \\ 10 & 7 \end{bmatrix}$$

If $A = B$, then solve for x, y, z

$$5x = 5 \qquad -z = 7$$

$$3x + y = 10 \qquad y = 7$$

$\Rightarrow x = 1, \; y = 7, \; z = -7$

Addition & Subtraction

If two matrices have the same dimensions then adding or subtracting matrices is done by adding or subtracting corresponding entries.

$$(A+B)_{ij} = A_{ij} + B_{ij}$$

$$\begin{bmatrix} 5 & 6 \\ 10 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 5+1 & 6+1 \\ 10+2 & 1+4 \end{bmatrix} = \begin{bmatrix} 6 & 7 \\ 12 & 5 \end{bmatrix}$$

Scalar Multiplication

We can multiply a matrix by a number. This scales every entry of the matrix.

$$5\begin{bmatrix} 10 & 12 \\ 13 & -1 \end{bmatrix} = \begin{bmatrix} 5\cdot 10 & 5\cdot 12 \\ 5\cdot 13 & 5\cdot(-1) \end{bmatrix} = \begin{bmatrix} 50 & 60 \\ 65 & -5 \end{bmatrix}$$

Transposition

The transpose of a matrix M, signified by M^T is a matrix where the rows of M^T are the columns of M.

$$M = \begin{bmatrix} 2 & 3 \\ 1 & 4 \\ 0 & -1 \end{bmatrix} \qquad M^T = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & -1 \end{bmatrix}$$

M is a 3x2 matrix $\qquad$ M^T is a 2x3 matrix

Note: $(A+B)^T = A^T + B^T$

$$(A^T)^T = A$$

If $A = A^T$, then we say that it is a symmetric matrix

Is $A = \begin{bmatrix} 2 & 3 \\ 3 & -1 \end{bmatrix}$ symmetric?

Yes because $A^T = \begin{bmatrix} 2 & 3 \\ 3 & -1 \end{bmatrix} = A$

Matrix Multiplication

Matrix multiplication differs from multiplication of numbers.

- The product of two matrices is not always defined
- $A \cdot B$ does not always equal $B \cdot A$

The multiplication matrices A and B is only defined if the number of columns of A equals the number of rows of B

If A is an $m \times n$ matrix
B is a $p \times q$ matrix

AB is only defined if

$n = p$

$A \cdot B$ is defined if $n = p$

$m \times n$ $\quad$ $p \times q$

must be equal

dimensions of result

and will have dimensions $m \times q$

A 3×2 matrix multiplied by 2×5 matrix will have the dimension 3×5

By this we see that the dimensions of a $1 \times n$ matrix multiplied by a $n \times 1$ matrix will be 1×1, which we can think of as just a number.

$$\begin{bmatrix} 2 & 3 \end{bmatrix} \begin{bmatrix} 7 \\ 1 \end{bmatrix} = 2 \cdot 7 + 3 \cdot 1 = 14 + 3 = 17$$

$1 \times 2 \cdot 2 \times 1$

$$\begin{bmatrix}5 & 3 & 2\end{bmatrix}\begin{bmatrix}1\\2\\4\end{bmatrix} = 5\cdot 1 + 3\cdot 2 + 2\cdot 4 = 5+6+8$$
$$= 19$$

$$\begin{bmatrix}1 & 1 & 2 & 3\end{bmatrix}\begin{bmatrix}5\\8\\0\\2\end{bmatrix} = 1\cdot 5 + 1\cdot 8 + 2\cdot 0 + 3\cdot 2$$
$$= 5 + 8 + 0 + 6 = 19$$

Matrix Multiplication Def.

The ij-th entry of AB is equal to the result of the i^{th} row of A multiplied by the j^{th} col of B.

$$\begin{pmatrix}1 & 2\\3 & 4\end{pmatrix}\cdot\begin{pmatrix}6 & 2\\0 & 1\end{pmatrix} = \begin{pmatrix}(1\ 2)\begin{pmatrix}6\\0\end{pmatrix} & (1\ 2)\begin{pmatrix}2\\1\end{pmatrix}\\(3\ 4)\begin{pmatrix}6\\0\end{pmatrix} & (3\ 4)\begin{pmatrix}2\\1\end{pmatrix}\end{pmatrix} = \begin{pmatrix}6 & 4\\18 & 10\end{pmatrix}$$

$$(1\ \ 2)\begin{pmatrix}6\\0\end{pmatrix} = 1\cdot 6 + 2\cdot 0 = 6$$

$$(1\ \ 2)\begin{pmatrix}2\\1\end{pmatrix} = 1\cdot 2 + 2\cdot 1 = 4$$

$$(3\ \ 4)\begin{pmatrix}6\\0\end{pmatrix} = 3\cdot 6 + 4\cdot 0 = 18$$

$$(3\ \ 4)\begin{pmatrix}2\\1\end{pmatrix} = 3\cdot 2 + 4\cdot 1 = 10$$

$$\begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 0 & -1 & 3 \\ -2 & 0 & 4 \end{pmatrix} = \begin{pmatrix} \ & \ & \ \end{pmatrix}$$

$$1\times 2 \qquad 2\times 3 \qquad = \begin{pmatrix} -4 & -1 & 11 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ -2 \end{pmatrix} = 1\cdot 0 + 2\cdot(-2) = -4$$

$$\begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \end{pmatrix} = 1(-1) + 2\cdot 0 = -1$$

$$\begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = 1\cdot 3 + 2\cdot 4 = 11$$

	people who have read the book	people who have seen the movie
Lord of the Rings	100	350
Princess Bride	125	250
To Kill a Mockingbird	200	150

Suppose 20% of people who have read the book own the book and 5% of the people who have seen the movie own the movie.

For each IP, how many people own either the book or movie?

(# who read it)(0.20) + (# who watched it)(0.05)

$$\begin{pmatrix} 100 & 350 \\ 125 & 250 \\ 200 & 150 \end{pmatrix}\begin{pmatrix} 0.20 \\ 0.05 \end{pmatrix} = \begin{pmatrix} 37.5 \\ 37.5 \\ 47.5 \end{pmatrix}$$

Just own the book

$$\begin{pmatrix} 100 & 350 \\ 125 & 250 \\ 200 & 150 \end{pmatrix}\begin{pmatrix} 0.20 \\ 0 \end{pmatrix} = \begin{pmatrix} 20 \\ 25 \\ 40 \end{pmatrix}$$

20 people own LOTR as a book

25 people own Princess Bride as a book

40 people own To Kill a Mockingbird as a book