Probability via Counting

When we talked about modelled probability, we said that the probality of an event E was

$$P(E) = \frac{n(E)}{n(S)} = \frac{\#\text{ of favorable outcomes}}{\#\text{ of total outcomes}}$$

So we can apply our methods of counting (decision algorithm, permutations, combinations) to calculate probabilities.

ex There are 2 red, 3 blue, and 4 green marbles. What is the probability of choosing 3 marbles and getting all the reds?

$$P\left(\begin{matrix}\text{all reds in}\\ \text{3 drawn}\end{matrix}\right) = \frac{n(\text{all reds})}{n(\text{choosing 3 marbles})} = \frac{7}{C(9,3)} = \frac{7}{84}$$

$$= \frac{1}{12}$$

all reds:

Step 1 choose 2 reds $C(2,2) = 1$

Step 2 choose 1 not red $C(7,1) = 7$

ex There are 2 red, 3 blue, and 4 green marbles. What is probability of choosing 3 marbles and getting at least 1 blue?

$$P(\geq 1 \text{ blue}) = \frac{n(\geq 1 \text{ blue})}{n(\text{choose 3 marbles})} = \frac{64}{84} = \frac{16 \cdot \cancel{4}}{21 \cdot \cancel{4}} = \frac{16}{21}$$

$n(\geq 1 \text{ blue})$

$= (3 \cdot 15) +$

$(3 \cdot 6) +$

$(1 \quad)$

$= 45 + 18 + 1$

$= 64$

alt 1: exactly 1 blue

step 1: choose 1 blue $C(3,1) = 3$

step 2: choose 2 not blue $C(6,2) = 15$

alt 2: exactly 2 blue

step 1: choose 2 blues $C(3,2) = 3$

step 2: choose 1 not blue $C(6,1) = 6$

alt 3: exactly 3 blue

step 1: choose 3 blue $C(3,3) = 1$

$$n(E) = n(S) - n(E')$$

$$n(\geq 1 \text{ blue}) = n(\text{choose 3 marbles}) - n(0 \text{ blues})$$

$$= 84 - C(6,3)$$

$$= 84 - \frac{\cancel{6} \cdot 5 \cdot 4}{\cancel{3} \cdot \cancel{2}} = 84 - 20 = 64$$

ex A pizza shop has 18 toppings, 6 of which are different meats. What's the probability that a random 2-topping pizza is vegetarian?

$$P\left(\begin{matrix}\text{2 topping pizza}\\ \text{vegetarian}\end{matrix}\right) = \frac{n(\text{2 topping vegetarian pizzas})}{n(\text{2 topping pizzas})}$$

$$= \frac{C(12,2)}{C(18,2)} = \frac{\frac{12!}{10!\ 2!}}{\frac{18!}{16!\ 2!}}$$

$$= \frac{12!\ 16!\ \cancel{2!}}{18!\ 10!\ \cancel{2!}} = \frac{12\cdot 11\cdot \cancel{10!}\cdot \cancel{16!}}{18\cdot 17\cdot \cancel{16!}\cdot \cancel{10!}}$$

$$= \frac{\overset{4}{\cancel{12}}\cdot 11}{\underset{6}{\cancel{18}}\cdot 17} = \frac{\overset{2}{\cancel{4}}\cdot 11}{\underset{3}{\cancel{6}}\cdot 17} = \frac{2\cdot 11}{3\cdot 17} = \frac{22}{51}$$

$$P\left(\begin{matrix}\text{2 topping}\\ \text{veggie pizza}\end{matrix}\right) = 22/51 \approx 0.43$$

Probabilities of Poker Hands

A poker hand consists of 5 cards from a 52 card deck. Each card has a suit — Hearts, Spades, Clubs, Diamonds — and a value — 2, 3, ..., 10, Jack, Queen, King, Ace.

- How many different poker hands are there?

$$C(52,5) = \frac{\overset{26}{\cancel{52}} \cdot \overset{17}{\cancel{51}} \cdot \overset{10}{\cancel{50}} \cdot 49 \cdot \overset{12}{\cancel{48}}}{\cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1} = 26 \cdot 17 \cdot 10 \cdot 49 \cdot 12$$

$$= 2{,}598{,}960$$

Some kinds of hands have special names:

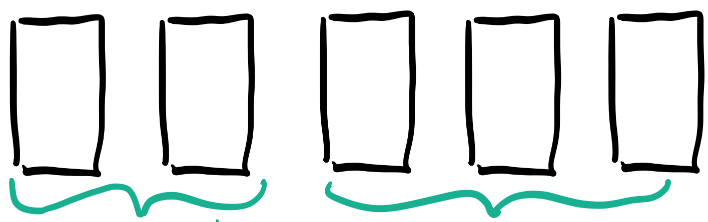
- 1 pair: exactly 2 cards share a value, no others share a value — $\underbrace{2S, 2H}, 9H, KC, 7S$
- 2 pairs: 2 sets of 2 cards sharing a value (2 sets are of different values), 5th card is a new value — $\underbrace{2S, 2H}, \underbrace{9H, 9D}, KC$
- 3 of kind — 3H, 3S, 3D, 6C, AD
- straight — 8S, 9C, 10C, JH, QD
- flush — 2H, 4H, 7H, QH, AH

- full house

⋮

2H, 2D, 7C, 7S, 7D

$$P(1\text{ pair}) = \frac{n(1\text{ pair})}{n(\text{hands})} = \frac{1{,}098{,}240}{2{,}598{,}960} = 0.4226$$



- same value
- suits do not repeat

- values do not repeat
- values not same as pair
- suits can repeat

step 1: value of pair $C(13,1) = 13$

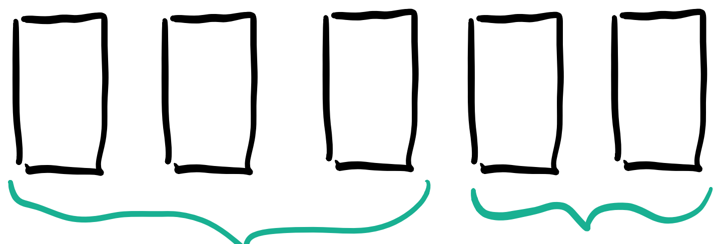
step 2: suits of pair $C(4,2) = 6$

step 3: values of remaining $C(12,3) = \frac{{}^{2}\cancel{12}\cdot 11\cdot 10}{\cancel{3}\cdot\cancel{2}} = 220$

step 4: suits of remaining $4\cdot 4\cdot 4 = 64$

$$n(1\text{ pair}) = 13\cdot 6\cdot 220\cdot 64 = 1{,}098{,}240$$

$$P(3 \text{ of a kind}) = \frac{n(3 \text{ of kind})}{n(\text{hands})} = \frac{54{,}912}{2{,}598{,}960} = 0.02113$$



- all same value
- suits won't repeat

- 2 different values, not value of 3 of kind
- suits can repeat

Step 1: value of 3 of kind $C(13,1) = 13$

Step 2: suits of 3 of kind $C(4,3) = 4$

Step 3: values of remaining 2 $C(12,2) = 66$

Step 4: suits of remaining 2 $4 \cdot 4 = 16$

$$n(3 \text{ of kind}) = 13 \cdot 4 \cdot 66 \cdot 16 = 54{,}912$$