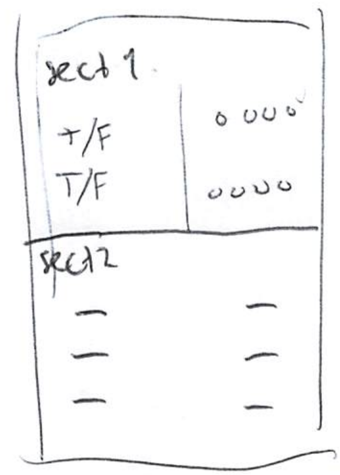
25 Mar

Refresher

A test has 2 sections. The first section allows students to either answer 2 T/F problems or 2 multiple choice (4 options). The second section is matching 3 definitions with 3 terms.
How many ways to fill out exam?



Step 1: section 1

alternative 1: T/F problems

$2\cdot 2 = 4$
- Step 1: answer 1st T/F : 2 options
- Step 2: answer 2nd : 2 options

alternative 2: multi choice

$4\cdot 4 = 16$
- Step 1: answer 1st : 4 options
- Step 2: answer 2nd : 4 options

$4+16 = 20$

Step 2: section 2

$3\cdot 2\cdot 1 = 6$
- Step 1: match 1st term : 3 options
- Step 2: match 2nd term : 2 options
- Step 3: match 3rd term : 1 option

$20 * 6 = 120$

* multiply steps / add alternatives

$$\overbrace{\Big(\underbrace{(2\cdot 2)}_{T/F} + \underbrace{(4\cdot 4)}_{Multi}\Big)}^{\text{section 1}} \cdot \overbrace{\Big((3\cdot 2\cdot 1)\Big)}^{\text{section 2}} = 120$$

6.4 Permutations and Combinations

How many ways can 5 students line up?

Using decision algorithm...

Step 1:	choose	1st in line	:	5 options
Step 2:	choose	2nd in line	:	4 options
Step 3:	choose	3rd	:	3 options
Step 4:	choose	4th	:	2
Step 5:	choose	5th	:	1

$5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ many ways

What if there were 10 students?

... $10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

Both of these look like

(number of people)(num. people −1)$\cdots$(2)(1)

To save space we use the notation '$n!$'
read n factorial.

$$n! = (n)(n-1)(n-2)\cdots(2)(1)$$

& $0! = 1$

Each way of arranging the students is called a <u>permutation</u> (an ordered list) of the students.

If we have n elements and ask how many permutations there are, the answer is

$$n!$$

However, sometimes we don't want to order all the items, only some.

Suppose there were 7 students and we asked how many ways for 3 to line up?

Step 1: Choose 1st : 7 options
Step 2: Choose 2nd : 6 options
Step 3: Choose 3rd : 5 options

So there are $7\cdot6\cdot5$ ways

notice that $$7\cdot6\cdot5 = \frac{7\cdot6\cdot5\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}{\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{1}} = \frac{7!}{4!}$$

which is very concise.
Where did 4 come from?
We wanted 3 students out of 7
and $(7-3)=4$

The number of permutations of r items taken from a total of n items is

$$\overbrace{(n)(n-1)(n-2)\cdots(n-(r-1))}^{r \text{ many terms}} = \frac{n!}{(n-r)!}$$

We might use $P(n,r)$ to # permutations of r items from a total of n

<u>note</u>: ~~Writing~~ Using factorials makes writing answers easy but sometimes computationally hard.

ex $\frac{123!}{121!}$ 123! and 121! too big for many calculators to compute

(123! is bigger than # of atoms in universe squared)

$$\frac{123!}{121!} = \frac{123 \cdot 122 \cdot \cancel{121} \cdot \cancel{120} \cdots}{\cancel{121} \cdot \cancel{120} \cdots} = 123 \cdot 122$$

- How many ways to shuffle a deck of cards? $52!$

- Number of ways to take 30 songs and make a 30 song playlist? $30!$

- Number of ways to make a playlist of 10 songs from 30?

$$30 \cdot 29 \cdot 28 \cdot 27 \cdot 26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 \cdot 21$$

$$= \frac{30!}{(30-10)!}$$

- Number of two topping pizzas if there are a total of 5 toppings?

$\frac{5!}{(5-2)!}$? is a hamburger pepperoni pizza different from a pepperoni hamburg pizza?

Remember <u>permutations</u> are ordered arrangements. The different order of pizza toppings ~~don't~~ doesn't make a different pizza.

Let our toppings be

H : hamburger
P : pepperoni
B : banana peppers
S : spinach
O : onions

We said theres $\frac{5!}{(5-2)!}$ permutations of 2 toppings

HP	HB	HS	HO	PB	PS	PO	BS	BO	SO
PH	BH	SH	OH	BP	SP	OP	SB	OB	OS

(note: finding # of perms is listing them all out
cons: tedious, easy to make mistakes)

Let's circle actually different pizzas

There are 10 2-topping pizzas.
How many permutations in each circle? $2 = 2!$

So # of 2 topping pizzas from 5 toppings $= \dfrac{\text{\# permutations of 2 items from 5}}{\text{\# permutations of 2 items from 2}}$

$$= \frac{\frac{5!}{(5-2)!}}{2!} = \frac{5!}{(5-2)!\,2!}$$

We are counting the number of unordered sets. These are called <u>combinations</u>.

in a <u>combination</u> order does not matter

↗ unordered sets

in a <u>permutation</u> order does matter

↳ ordered lists

A combination of r items from a total n items is

$$C(n,r) = \frac{P(n,r)}{r!} = \frac{n!}{(n-r)!\,r!}$$

★ there are more permutations than combinations ★

There are 30 songs to choose from. You only listen to music on shuffle. How many 10 song playlists?

- with shuffle turned off then order of playlist has meaning

$\Rightarrow$ permutation

$$\Rightarrow \cancel{P(10,3) = 10\cdot 9\cdot 8 = \frac{10!}{(10-3)!}}$$

$$P(30,10)$$

- with shuffle turned on different orderings of the same songs doesn't change anything

$\Rightarrow$ combination

$$\Rightarrow \cancel{C(10,3) = \frac{P(10,3)}{3!} = \frac{10\cdot 9\cdot 8}{3\cdot 2\cdot 1}}$$

$$C(30,10) = \cancel{\frac{10!}{(10-3)!\,3!}}$$

10 Students try out for the play Romeo & Juliet.

- number of ways to assign Romeo, Juliet, Mercutio?

→ order matters, list not a set

→ Permutation, $P(10,3) = 10 \cdot 9 \cdot 8$

- number of ways to assign 3 wenches

→ set of actors, not list

→ combination

$$C(10,3) = \frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1}$$

At a ~~pet~~ shelter there are 7 dogs and 6 cats. A family comes in hoping to adopt 2 cats and 3 dogs. How many possible adoptions possible?

- asking about sets not lists
- unordered ⇒ combinations

cats ~~that can~~ possibilities

$C(6,2)$

dog possibilities

$C(7,3)$

→ because we want cats and dogs, not one or the other we multiply

$$C(6,2) * C(7,3)$$

$$= \frac{7!\ 6!}{(7-3)!3!(6-2)!\ 2!} = 525$$

$$\frac{7 \cdot \cancel{6} \cdot 5 \cdot \cancel{4 \cdot 3 \cdot 2 \cdot 1} \cdot 6 \cdot 5 \cdot \cancel{4 \cdot 3 \cdot 2 \cdot 1}}{\cancel{4 \cdot 3 \cdot 2 \cdot 1}\ \cancel{3} \cdot 2 \cdot 1\ \ \cancel{4 \cdot 3 \cdot 2 \cdot 1} \cdot 2 \cdot 1}$$

$$= \frac{7 \cdot 5 \cdot 6 \cdot 5}{2}$$

You have 4 red marbles
3 green marbles
3 cateyes marbles
1 pink marble

A friend wants to buy 4 marbles but only if they are each different color. How many ways to sell some of your marbles?

any sold set of marbles contains
1 red, 1 green, 1 cateyes, 1 pink marble

$$C(4,1) = 4 \qquad C(3,1) = 3 \qquad C(3,1) = 3 \qquad C(1,1) = 1$$

$$4 \cdot 3 \cdot 3 \cdot 1 = 36$$

How many 2 ~~top~~ or 3 topping pizzas include mushrooms?

Toppings = {M, P, H, B, S}

2 topping contain mushroom

$C(4,1)$

$= 4$

3 topping w/ mushrooms

$C(4,2)$

$= \frac{4 \cdot 3}{2} = 6$

<u>or</u> signifies these are alternatives rather than steps

$4 + 6 = 10$