Refresher

You and a friend play a game where you each choose a number 1 – 5. If the sum is even, you win ~~by~~ the sum # of pts. If ~~you~~ the sum is odd, you lose the sum # of pts. Fill in the payoff matrix P.

$$
\begin{array}{cc}
 & \text{friend} \\
 & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\
\text{you} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} 2 & -3 & 4 & -5 & 6 \\ -3 & 4 & -5 & 6 & -7 \\ 4 & -5 & \ddots & & \\ -5 & 6 & & & \\ 6 & -7 & & & \end{bmatrix}
\end{array}
$$

4.4 cont

20 Feb

Solving a game involves finding the optimal strategy for each player.

The expected value of a game is the expected payoff when both players use an optimal strategy.

The EV of G has the property that:

- if the row player plays optimally, then the expected payoff will always $\geq$ EV of G
- if the col player plays optimally, then the expected payoff will always $\leq$ EV of G

Because of this, some larger game can be solved by just looking at the worst case scenario of each pure strategy.

Consider the following game:

$$
\begin{array}{ccc}
 & & B \\
 & & \begin{array}{ccc} p & q & r \end{array} \\
A & \begin{array}{c} s \\ t \\ u \end{array} & \begin{pmatrix} \boxed{-4} & -3 & 3 \\ 2 & -1 & \boxed{-2} \\ 1 & \boxed{0} & 2 \end{pmatrix}
\end{array}
$$

- For each pure strategy of A, what is the worst case scenario?
  - the <u>row minima</u> are the worst case scenarios
  - let's put □ around each row minima
- For each pure strategy of B, what is the worst case scenario?
  - the <u>col. maxima</u> are the worst case scenarios
  - let's put △ around each col maxima

- By playing only u, A can force EP ≥ 0
- By playing only q, B can force EP ≤ 0

- Thus the EV of G must be 0 and the above pure strategies are optimal

If a row minima is also a col maxima, then it is called a <u>saddle point</u> and it is the expected value of the game.

The optimal strategies are the pure strategy of choosing the row containing the saddle pt and choosing the col containing the saddle pt.

ex The payoff matrix for TV corp A playing a movie, documentary, or sitcom vs TV corp B playing reality show, cspan, local news is

$$
\begin{array}{cc}
 & B \\
 & \begin{array}{ccc} rs & cspan & news \end{array} \\
A \quad \begin{array}{c} mov \\ doc \\ sit \end{array} & \begin{pmatrix} \boxed{-1} & 4 & 1 \\ -1 & \boxed{-2} & 2 \\ \boxed{-2} & 3 & 0 \end{pmatrix}
\end{array}
$$

What are the optimal strategies for A and B?

- Is there a saddle pt?

  □ row minima, △ col maxima

  * a saddle pt exists, A should play movie, B should play reality show.

- Does it reduce? do any rows dominate another? do any cols dominate another?

$$
\begin{pmatrix} -1 & 4 & 1 \\ -1 & -2 & 2 \\ -2 & 3 & 0 \end{pmatrix}
$$

1. $-1 \geq -2$, $4 \geq 3$, $1 \geq 0$ so row 1 dominates row 3
2. $-1 \leq 1$, $-1 \leq 2$ so col 1 dominates col 3
3. $-1 \geq -1$, $4 \geq -2$ so row 1 dom. row 2
4. $-1 \leq 4$, so col 1 dom col 2

* reduces to just A plays movie, B plays reality; these is optimal

EV of G is -1

ex McDonalds and BK are competing for sales. The payoff matrix for bringing in customers with McD's promotions a, b, c and BK deals d, e, f is

$$
\begin{array}{ccc}
 & & \text{BK} \\
 & & \begin{array}{ccc} d & e & f \end{array} \\
\text{McD} & \begin{array}{c} a \\ b \\ c \end{array} & \begin{pmatrix} 3 & \boxed{-3} & -2 \\ 2 & 2 & \boxed{1} \\ \boxed{-1} & 3 & 0 \end{pmatrix}
\end{array}
$$

What is the EV of G?

- does this reduce? carefully checking reveals that no row/col dominates another

- Is there a saddle point?
  - □ row minima
  - △ col maxima

* there exists a saddle pt,
  - McD should run promo b
  - BK should run deal f
  - EV of G 1

Can we solve something that doesn't reduce to 1x1 nor has a saddle pt?

Yes, because every counter strategy is pure, we can look at each one.

Recall the example with Waffle House and Ihop that reduced to

$$
\begin{array}{ccc}
 & & I \\
 & & \begin{array}{cc} H & C \end{array} \\
WH & \begin{array}{c} H \\ A \end{array} & \begin{pmatrix} -1 & 1 \\ 3 & -1 \end{pmatrix}
\end{array}
$$

First find optimal strategy for WH

- $R = [x \quad y]$ $x + y = 1$ so $y = 1 - x$

  $= [x \quad \cancel{1-x}]$

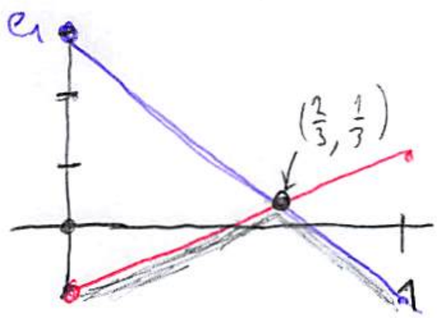
  $= [x \quad 1-x]$

- Ihop's counter strategy is pure, there's 2 of them

$$\begin{array}{cccc} & R & P & C \end{array}$$

$$e_1 = [x \quad 1-x]\begin{bmatrix} -1 & 1 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = [x \quad 1-x]\begin{bmatrix} -1 \\ 3 \end{bmatrix} = -x + 3(1-x) = 3 - 4x$$

$$e_2 = [x \quad 1-x]\begin{bmatrix} -1 & 1 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = [x \quad 1-x]\begin{bmatrix} 1 \\ -1 \end{bmatrix} = x - (1-x) = 2x - 1$$



- To minimize the maximum Ihop damage WH should choose where these lines intersect.

so $x = \frac{2}{3}$ WH strategy is $\left[\frac{2}{3} \quad \underset{\substack{\uparrow \\ 1-x}}{\frac{1}{3}}\right]$

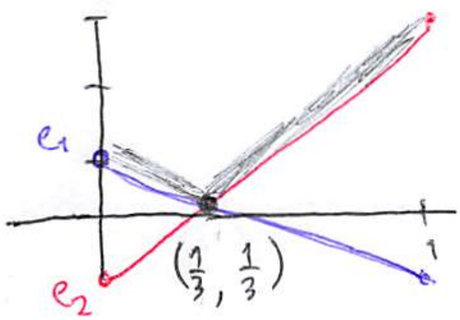
Let's find the optimal strategy for Ihop

$\bullet\ C = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 1-x \end{bmatrix}$

• WH counter strategies are pure, let's look at both

$e_1 = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x \\ 1-x \end{bmatrix} = \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ 1-x \end{bmatrix} = -x + 1 - x = 1-2x$

$e_2 = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x \\ 1-x \end{bmatrix} = \begin{bmatrix} 3 & -1 \end{bmatrix} \begin{bmatrix} x \\ 1-x \end{bmatrix} = 3x - (1-x) = 4x-1$



$1 - 2x = 4x - 1$
$2 = 6x$
$x = 1/3$

$1 - 2(1/3) = 1/3$
$4(\frac{1}{3}) - 1 = 1/3$

So optimal strategy is $x = \frac{1}{3}$ or $\begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$

If we find EV of G

$\begin{bmatrix} 2/3 & 1/3 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix} = 1/3$

Recap

- First find optimal strategy for each player
- to find optimal strategy
  - defined ~~it~~ in terms of x
  - Construct a line for each pure counter strategy
  - choose x that minimizes (for row) or maximizes (for col) the maximum damage; find the x-coord of where the two lines intersect

Prisoner's Dilemma

Imagine 2 thieves caught and interrogated. They're informed of the following:

- if neither inform on the other, both get a 1 yr sentence.
- if 1 informs on the other, the informant goes free and the other has a 3 yr sentence
- if both inform on each other, 2 yr sentence for both

A \ B	B stays silent	B informs
A stays silent	-1 -1	0 -3
A informs	-3 0	-2 -2

- it works out that the 'rational' decision for each thieve is to betray

* is cooperation unprofitable?

* if this is repeated and played multiple times

'always betray' is no longer optimal

explore ncase.me/test