Refresher

Let $A = \begin{bmatrix} 2 & -1 & -1 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$. Does $A^{-1} = \begin{bmatrix} 1 & -1 & 2 \\ 0 & -1 & 1 \\ 1 & -1 & 3 \end{bmatrix}$?

- $A A^{-1} = I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

- $\begin{pmatrix} 2 & -1 & -1 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 2 \\ 0 & -1 & 1 \\ 1 & -1 & 3 \end{pmatrix} = \begin{pmatrix} & & \\ & \underbrace{0} & \\ & & \end{pmatrix} \neq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$2 \cdot (-1) + 0(-1) + (-2)(-1) = 0$$

$$A^{-1} \neq \begin{pmatrix} 1 & -1 & 2 \\ 0 & -1 & 1 \\ 1 & -1 & 3 \end{pmatrix}$$

2/18

4.4 Game Theory

In rock, paper, scissors, rock beats scissors, scissors beats paper, and paper beats rock. Between two players A and B we can represent all scenarios as follows

What B chooses

What A chooses	rock	paper	scissors
rock	tie	A loses	A wins
paper	A wins	tie	A loses
scissors	A loses	A wins	tie

if we replace: ties with 0
A wins with +1
A loses with -1

$$\begin{array}{c c} & \begin{array}{ccc} r & p & s \end{array} \\ \begin{array}{c} r \\ p \\ s \end{array} & \begin{bmatrix} 0 & -1 & +1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \end{array}$$

We could express

"player A chose rock" as $\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$

and "player B chose scissors" as $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

the outcome of these choices is

$$\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 1$$

a win for player A

We could express

"player A chooses rock 50% and scissors 50%"

$$\begin{bmatrix} 0.5 & 0 & 0.5 \end{bmatrix}$$

"player B chooses rock 50% and paper and scissors 25%" $\begin{bmatrix} .5 \\ .25 \\ .25 \end{bmatrix}$

the average outcome after many games is

$$\begin{bmatrix} 0.5 & 0 & 0.5 \end{bmatrix} \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} .5 \\ .25 \\ .25 \end{bmatrix}$$

$$= \begin{bmatrix} -0.5 & 0 & 0.5 \end{bmatrix} \begin{bmatrix} .5 \\ .25 \\ .25 \end{bmatrix} = -.25 + 0.125 = -0.125$$

$$= -1/8$$

Player A loses more frequently. About every 8 games, Player A will have lost 1 more game than they won.

This is an example of a <u>two-person zero sum game</u>, where one player's loss is the other player's gain.

If ~~[illegible]~~ player A has m choices of moves and player B has n choices, then the $m \times n$ matrix that shows the result of each possible scenario is the <u>payoff matrix</u> P.

<u>ex</u> in previous example

$$P = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

We use positive entries as being favorable for player A, the row player, and negative entries as unfavorable for A.

The opposite of this is true for player B, the column player.

How a player chooses a move is called its <u>strategy</u>. A player using the same move every time is called a <u>pure strategy</u>. Choosing moves for certain percents of the time in a random fashion is called a <u>mixed strategy</u>.

<u>ex</u> "Always chose rock" is a pure strategy

"Choose rock 50%, paper 50%" is a mixed strategy

The <u>expected payoff</u> is the result of a pair of strategies.

Given a payoff matrix P, the strategy R of the row player, and the strategy C of the column player, then

$$\text{expected payoff} = e = R \cdot P \cdot C$$

- the row player wants to maximize e
- the column player wants to minimize e

Waffle House and IHop are each planning a new location in 1 of 3 possible areas:
Hillsborough St, Cameron Village, Avent Ferry.
The success of each location depends on where the other business chooses. The following payoff matrix models the situation, where each point is a shift of 1,000 monthly customers.

		IHop H	IHop C	IHop A
WH	H	-1	1	2
	C	-2	0	1
	A	3	-1	1

problem 1: If WH learns that Ihop is going to choose Hills. or Cam. and these two locations are equally likely, where should WH locate?

- we know Ihop's strategy $\begin{bmatrix} .5 \\ .5 \\ 0 \end{bmatrix}$
- we don't know WH's strategy, let's call it $\begin{bmatrix} x & y & z \end{bmatrix}$

$x+y+z=1, \quad x, y, z \geq 0$

- expected payoff $= e = R \cdot P \cdot C = \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} -1 & 1 & 2 \\ -2 & 0 & 1 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} .5 \\ .5 \\ 0 \end{bmatrix}$

$$= \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = -y + z$$

- WH wants to maximize e, maximize $-y+z$

  set $y=0,\ z=1,\ x=0$

  so $\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$

- WH should build at Avent Ferry

Problem 2: Suppose Ihop learns that WH is going to build in Cameron Village. Where should Ihop locate?

- We know WH's strategy $\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$
- We don't know I's, call $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$
- expected payoff $= R \cdot P \cdot C = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 1 & 2 \\ -2 & 0 & 1 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

  $$= \begin{bmatrix} -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = -2x + z$$

- Ihop is the column player, Ihop wants to minize e so minize $-2x+z$
- $x=1,\ y=0,\ z=0$ $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ I should Build at Hillsborough

Problem 3: Are there any locations that WH or Ihop should never plan on building at?

- From WH's point of view (wants bigger #'s) building on Hills is "better" than building on Cam. for each possible I hop location.

$-1 \geq -2$ (if I chooses H) $\quad 1 \geq 0$ (if I chose C) $\quad 2 \geq 1$ (if I chose A)

- From Ihops' POV (wants smaller #'s), Cam. is "better" than building on Avent. in all Scenarios.

$1 \leq 2$ (WH chooses H) $\quad 0 \leq 1$ (... C) $\quad -1 \leq 1$ (... A)

Our game/scenario reduces to just

	H	C
H	-1	1
A	3	-1

In general:

- row i <u>dominates</u> row j if each entry in row i is greater than or equal to the corresponding entry in row j

- col i <u>dominates</u> col j if each entry in col i is less than or equal to the corresponding entry in col j

To reduce a payoff matrix by dominance:

1. Check if any row dominates another; remove the dominated rows
2. " " cols " " ; "
   " Cols
3. Repeat 1 and 2 until no dominated rows or columns

Solving a 2x2 game:

If you choose a mixed strategy your opponent can find an appropriate pure counterstrategy. Thus an optimal strategy is one that minimizes the maximum damage an opponent can cause. This is called the minimax criterion.

Here we assume each player tries to use its best strategy and assumes the other player is doing the same.