Let $A = \begin{pmatrix} 5 & x \\ y & 7 \end{pmatrix}$ $B = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$ $C = \begin{pmatrix} x & 0 \\ -1 & 2 \end{pmatrix}$

Find $(A * B) - C$

$$A * B = \begin{pmatrix} 5 & x \\ y & 7 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 5 - x & 10 + 3x \\ y - 7 & 2y + 21 \end{pmatrix}$$

$2 \times 2 \quad 2 \times 2$

$$(A * B) - C = \begin{pmatrix} 5 - x & 10 + 3x \\ y - 7 & 2y + 21 \end{pmatrix} - \begin{pmatrix} x & 0 \\ -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 5 - 2x & 10 + 3x \\ y - 6 & 2y + 19 \end{pmatrix}$$

Recall that we have defined matrix multiplication, which allows us to make statements like

$$AX = B$$

where A, X, and B are matrices.

- if we knew A and X but not B, we can find B with just matrix multiplication.

- if we knew A and B but not X

  - if X and B are column vectors, that's exactly what we've been studying

$$\underbrace{\begin{bmatrix} 0 & 2 & 3 \\ 1 & 4 & 5 \\ 0 & 6 & 5 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix}}_{B} \iff \begin{aligned} 0x + 2y + 3z &= 3 \\ 1x + 4y + 5z &= 5 \\ 0x + 6y + 5z &= 6 \end{aligned}$$

we've studied this

even if they're not column vectors, we can still use our tools

ex

$$\underbrace{\begin{bmatrix} 5 & 3 \\ 2 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x & y \\ z & w \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 2 & 7 \\ 1 & 2 \end{bmatrix}}_{B}$$

$$\Leftrightarrow \begin{aligned} 5x + 3z &= 2 \\ 5y + 3w &= 7 \\ 2x + 1z &= 1 \\ 2y + 1w &= 2 \end{aligned} \qquad \Leftrightarrow \begin{aligned} 5x + 0y + 3z + 0w &= 2 \\ 0x + 5y + 0z + 5w &= 7 \\ &\vdots \end{aligned}$$

* reducing the augmented matrix $(A \,\vdots\, B)$ here $\left(\begin{array}{cc|cc} 5 & 3 & 2 & 7 \\ 2 & 1 & 1 & 2 \end{array}\right)$

So good news; our techniques can be used to 'solve for X' in the matrix equation

$$AX = B$$

So what's the problem?

Different B matrix are treated as completely different problems.

Solving
$$5x+3y=1$$
$$7x-y=2$$
$$\begin{bmatrix}5 & 3\\7 & -1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix}$$
doesn't help us solve
$$5x+3y=1$$
$$7x-y=1$$
$$\begin{bmatrix}5 & 3\\7 & -1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}$$

If we were given the problem

$$A=\begin{bmatrix}5 & 3 & 2\\1 & 0 & 5\\-1 & 2 & 3\end{bmatrix},\quad X=\begin{bmatrix}x\\y\\z\end{bmatrix},\quad B_1=\begin{bmatrix}5\\3\\2\end{bmatrix},\quad B_2=\begin{bmatrix}-1\\0\\-1\end{bmatrix},\quad B_3=\begin{bmatrix}3\\-1\\1\end{bmatrix}$$

and we were tasked with finding X_1, X_2, X_3 s.t.

$$AX_1=B_1,\quad AX_2=B_2,\quad AX_3=B_3$$

We wouldn't be able to 'reuse' any of our calculations even though A is the same in each problem.

Here finding A^{-1} (inverse of A) makes our lives much easier

Important things to recall:

- the inverse of the $n \times n$ matrix A is the $n \times n$ matrix A^{-1} such that

$$AA^{-1} = A^{-1}A = I = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots & \\ & & & 1 \end{bmatrix}$$

- $IB = BI = B$

- If $A^{-1} = B$
then
$$BA = A^{-1}A = 1 = AA^{-1} = AB$$
so $$AB = BA = 1$$
so $$B^{-1} = A$$
restated: $A^{-1} = B$ means $B^{-1} = A$
or $(A^{-1})^{-1} = A$

If we found A^{-1} then

$$AX_1 = B_1 \quad\Downarrow\quad A^{-1}AX_1 = A^{-1}B_1 \quad\Downarrow\quad X_1 = A^{-1}B_1$$

$$AX_2 = B_2 \quad\Downarrow\quad A^{-1}AX_2 = A^{-1}B_2 \quad\Downarrow\quad X_2 = A^{-1}B_2$$

$$AX_3 = B_3 \quad\Downarrow\quad A^{-1}AX_3 = A^{-1}B_3 \quad\Downarrow\quad X_3 = A^{-1}B_3$$

So if we know A^{-1} then solving a system with A is done with just matrix multiplication!

<u>ex</u> Find X_1, X_2 where $AX_1 = B_1$ and $AX_2 = B_2$ where

$$A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \qquad B_1 = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \qquad B_2 = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$$

Old method: reduce $\left[\begin{array}{cc|c} 2 & 5 & 2 \\ 1 & 3 & 4 \end{array}\right]$ and interpret

reduce $\left[\begin{array}{cc|c} 2 & 5 & 0 \\ 1 & 3 & -3 \end{array}\right]$ and interpret

new method: first find A^{-1}.

I claim $A^{-1} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$

I can check by seeing $AA^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 6-5 & -10+10 \\ 3-3 & -5+6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

$$X_1 = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} B_1 = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} -14 \\ 6 \end{bmatrix}$$

$$X_2 = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} B_2 = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ -3 \end{bmatrix} = \begin{bmatrix} 15 \\ -6 \end{bmatrix}$$

So now we ask how to find A^{-1}

Finding A^{-1} is the same as solving the equality

$$AX = I$$

we can do this by reducing the augmented matrix

$$(A \,\vdots\, I)$$

ex $A = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$ find A^{-1}

- ie solving $AX = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
- reducing $\left[\begin{array}{cc:cc} 3 & 2 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{array}\right] \xrightarrow{R_1 := R_1 - R_2} \left[\begin{array}{cc:cc} 1 & 1 & 1 & -1 \\ 2 & 1 & 0 & 1 \end{array}\right]$

$$\xrightarrow{R_2 := R_2 - 2R_1} \begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & -1 & -2 & 3 \end{bmatrix} \xrightarrow{R_1 := R_1 + R_2} \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & -1 & -2 & 3 \end{bmatrix}$$

$$\xrightarrow{R_2 := -R_2} \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & -3 \end{bmatrix}$$ so $A^{-1} = \begin{bmatrix} -1 & 2 \\ 2 & -3 \end{bmatrix}$

$= I$

For 2×2 matrices, there's a nice formula to calculate inverses

if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

ex

$A = \begin{bmatrix} 2 & 5 \\ 7 & -1 \end{bmatrix}$ $A^{-1} = \frac{1}{-2-35}\begin{pmatrix} -1 & -5 \\ -7 & 2 \end{pmatrix} = \begin{pmatrix} 1/37 & 5/37 \\ 7/37 & -2/37 \end{pmatrix}$

$A = \begin{bmatrix} 3 & 4 \\ 4 & 6 \end{bmatrix}$ $A^{-1} = \frac{1}{18-16}\begin{pmatrix} 6 & -4 \\ -4 & 3 \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ -2 & 3/2 \end{pmatrix}$

non ex

$A = \begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix}$ $A^{-1} = \frac{1}{6-6}\begin{pmatrix} 1 & -2 \\ -3 & 6 \end{pmatrix} = \frac{1}{0}$

not good

Recall that some systems didn't have unique solutions (either redundant or inconsistent / inf. sol or 0 solutions).

Similarly, some matrices are noninvertible.

A noninvertible matrix is <u>singular</u>.

For 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

A is singular (non invertible) if and only if $ad - bc = 0$

In the 2×2 case $ad - bc$ is the <u>determinant</u> of A

In general A is singular iff it's determinant is 0.

<u>ex</u>

$$\det\left(\begin{pmatrix} 6 & 2 \\ 3 & 1 \end{pmatrix}\right) = (6)(1) - (2)(3) = 0$$

so no inverse for $\begin{pmatrix} 6 & 2 \\ 3 & 1 \end{pmatrix}$

we can try to reduce

$$\left(\begin{array}{cc|cc} 6 & 2 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{array}\right) \xrightarrow{\cdots} \left(\begin{array}{cc|cc} 3 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right)$$

row of zero's means we can't reduce it to $\left(\begin{array}{cc|c} 1 & 0 & \text{stuff} \\ 0 & 1 & \end{array}\right)$

Using Sage

inputting a matrix:

"A = matrix([[contents of row 1],[cont. row 2],...])"

reduce a matrix:

to reduce matrix B
first input the matrix B
then

"B.rref()"

inverse of a matrix:

input the matrix A

"A.inverse()"

algebra:

as expected
* note: write out implicit multiplication
do: "5*A" not "5A"
do: "A*(B-C)" not "A(B-C)"

transpose:

"A.transpose()" or "A.T"